

# Homework April 16th Some Solutions

## Problem 13

Is  $h : \mathbb{Q} \rightarrow \mathbb{Q}$ ,  $h(x) = 3x + 1$  onto? Justify your answer.

### Idea or thought process:

The function  $h$  multiplies the input  $x$  by 3, then adds 1. To reverse  $h$ , we subtract 1, then divide by 3. This process takes the codomain element  $y$  (which we assume to be rational number) to  $\frac{y-1}{3}$  (which is again a rational number so is in the domain of  $h$ ), so we expect the function  $h$  to still be onto.

### Solution:

Let  $y \in \text{codomain}(h) = \mathbb{Q}$ . Then  $y = h(x) = 3x + 1$  can be solved for  $x$  to get  $x = \frac{y-1}{3}$ . Since  $y$  is a rational number (fraction or ratio of integers),  $y - 1$  and  $(y - 1)/3$  are still rational numbers, so  $x = \frac{y-1}{3} \in \mathbb{Q} = \text{domain}(h)$ . Since we showed every  $y$  in the codomain is an output of some  $x$  in the domain, the function  $h$  is an onto function.

## Problem 14

Is  $j : \mathbb{Z} \rightarrow \mathbb{Q}$ ,  $j(x) = x/6$  onto? Justify your answer.

### Idea or thought process:

The function  $h$  divides the input  $x$  by 6. But the domain consists of integers  $\dots, -2, -1, 0, 1, 2, \dots$  so the outputs of  $h$  must be fractions whose denominators are  $\leq 6$ . So we expect this function to *not* be onto. We will take, say,  $y = 1/7$  as an element to test

### Solution:

We show  $1/7 \in \text{codomain}(j) = \mathbb{Q}$  is not an output of  $h$ . Let  $1/7 = j(x) = x/6$ . Solving for  $x$  gives  $x = 6/7 \notin \mathbb{Z} = \text{domain}(j)$ . So  $1/7$  is not an output of  $h$ . So  $h$  is not onto.

## Problem 18

Let  $j : \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$  be the function defined by  $j(x) = 1/x$ . Is  $j$  one-to-one?

### Idea or thought process:

Plug in some numbers to see whether  $j$  might be one-to-one or not:  $j(1) = 1$ ,  $j(2) = 1/2$ ,  $j(-1) = -1$ ,  $j(-2) = -1/2$ ,  $j(1/2) = 2$ . So these output are all different, we suspect  $j$  might be one-to-one. Let's check this.

### Solution:

To check that  $j$  is one-to-one: suppose inputs  $x$  and  $y$  gave the same output, so we write:

$$\begin{aligned} j(x_1) &= j(x_2) \\ \Rightarrow \frac{1}{x_1} &= \frac{1}{x_2} && \text{(Expand definition of } j) \\ \Rightarrow x_1 &= x_2 && \text{(Flip both fractions)} \end{aligned}$$

Since having equal outputs  $j(x) = j(x_2)$  forces the inputs  $x_1, x_2$  to be equal, we conclude  $j$  is a one-to-one function.

## Problem 21

Let  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$  be the function defined by  $f(x) = 1/x$ . Is  $f$  onto?

**Answer.**

To check that  $f$  is onto, suppose  $y$  is in the codomain of  $f$ , so  $y$  is a nonzero real number.

If  $f$  is onto, then  $y$  must be an output of  $f$  for some input  $x$ , so let's see what  $x$  is a candidate input: we set  $y = f(x) = 1/x$ . Solving for  $x$  by flipping fractions gives  $x = 1/y$ . So  $y$  is the output of  $f$  when the input is chosen to be  $1/y$ . So  $f$  is an onto function.

**Note:** We can check our answer:  $f(1/y) = \frac{1}{1/y} = y$ .

## Problem 22

Let  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = (x + 1)/x$ . Is  $f$  one-to-one?

**Answer.**

To check that  $f$  is one-to-one, suppose inputs  $x$  and  $x_2$  gave the same output:  $f(x_1) = f(x_2)$ , so

$$\begin{aligned}\frac{x_1 + 1}{x_1} &= \frac{x_2 + 1}{x_2} \\ \Rightarrow 1 + \frac{1}{x_1} &= 1 + \frac{1}{x_2} && \text{(Simplify fractions)} \\ \Rightarrow \frac{1}{x_1} &= \frac{1}{x_2} && \text{(Subtract 1)} \\ \Rightarrow x_1 &= x_2 && \text{(Take reciprocals)}\end{aligned}$$

Since equal outputs imply that the corresponding inputs are also equal, we deduce that  $f$  is a one-to-one function.

## Problem 23

Let  $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = (x + 1)/x$ . Is  $f$  onto?

**Answer.**

Observe that  $f(x) = \frac{x + 1}{x} = 1 + \frac{1}{x}$  so we suspect that 1 cannot be an output. Indeed, if we set

$1 = f(x) = 1 + \frac{1}{x}$ , we get  $0 = \frac{1}{x}$  and so  $0 \cdot x = 1$  or  $0 = 1$ , which can never occur. So there does not exist an  $x \in \mathbb{R} - \{0\}$  such that  $f(x) = 1$ . So  $f$  cannot be onto.